# 6.2 Linear and Almost Linear Sys (continued)

last time: $x' = F(x,y)$

$y' = G(x,y)$

is almost/locally linear if near critical pt $(x_0, y_0)$

it behaves like

$\vec{x}' = A\vec{x} + \vec{g}(\vec{x})$ such that $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \frac{|\vec{g}|}{|\vec{x}|} = 0$

generally, there is a different $A$ near each critical pt

example from last time:

$x' = -x + xy$

$y' = -2y + 8xy$ has a critical pt at $(\frac{1}{4}, 1)$

define $u = x - \frac{1}{4}$ $\quad v = y - 1$ $\qquad x = u + \frac{1}{4}$ $\quad y = v + 1$

$u' = x'$ $\quad v' = y'$

rewrite system

$$u' = -(u + \tfrac{1}{4}) + (u + \tfrac{1}{4})(v+1) = \tfrac{1}{4}v + uv$$

$$v' = \cdots = 8u + 8uv$$

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \frac{1}{4} \\ 8 & 0 \end{bmatrix}}_{A \text{ near } (\frac{1}{4}, 1)} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} uv \\ 8uv \end{bmatrix}$$

there is a different (more efficient) way to find the A matrix

→ linearize the nonlinear system

$$x' = F(x, y)$$

$$y' = G(x, y)$$

do Taylor series expansion for F and G near critical pt $(x_0, y_0)$

$$F(x,y) = \cancel{F(x_0,y_0)} + \frac{\partial F}{\partial x}(x_0,y_0)\overbrace{(x-x_0)}^{u} + \frac{\partial F}{\partial y}(x_0,y_0)\overbrace{(y-y_0)}^{v} + \cdots$$

higher order (not important)

$$G(x,y) = \cancel{G(x_0,y_0)} + \frac{\partial G}{\partial x}(x_0,y_0)\underbrace{(x-x_0)}_{u} + \frac{\partial G}{\partial y}(x_0,y_0)\underbrace{(y-y_0)}_{v} + \cdots$$

crit. pt.

near $(x_0,y_0)$ $\quad x' = F(x,y)$, $\; y' = G(x,y)$ $\quad$ becomes

$$u' = \frac{\partial F}{\partial x}(x_0,y_0)\,u + \frac{\partial F}{\partial y}(x_0,y_0)\,v$$

$$v' = \frac{\partial G}{\partial x}(x_0,y_0)\,u + \frac{\partial G}{\partial y}(x_0,y_0)\,v$$

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \underbrace{\begin{bmatrix} F_x(x_0,y_0) & F_y(x_0,y_0) \\ G_x(x_0,y_0) & G_y(x_0,y_0) \end{bmatrix}}_{\text{Jacobian matrix}} \begin{bmatrix} u \\ v \end{bmatrix}$$

Jacobian matrix $\rightarrow$ our $A$ matrix near $(x_0,y_0)$

example

$x' = x^2 + y^2 - 6 = F$

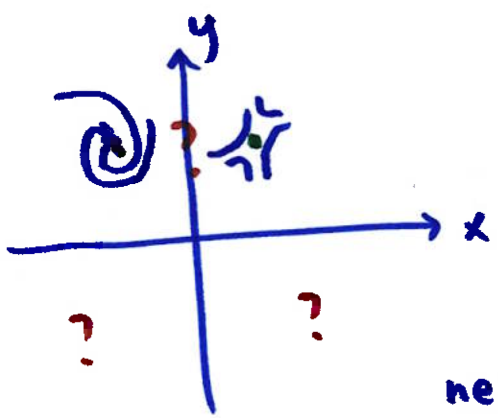
$y' = x^2 - y = G$

cp: $(\sqrt{2}, 2)$, $(-\sqrt{2}, 2)$

Jacobian matrix: $J(x,y) = \begin{bmatrix} 2x & 2y \\ 2x & -1 \end{bmatrix}$

linearized near $(\sqrt{2}, 2)$

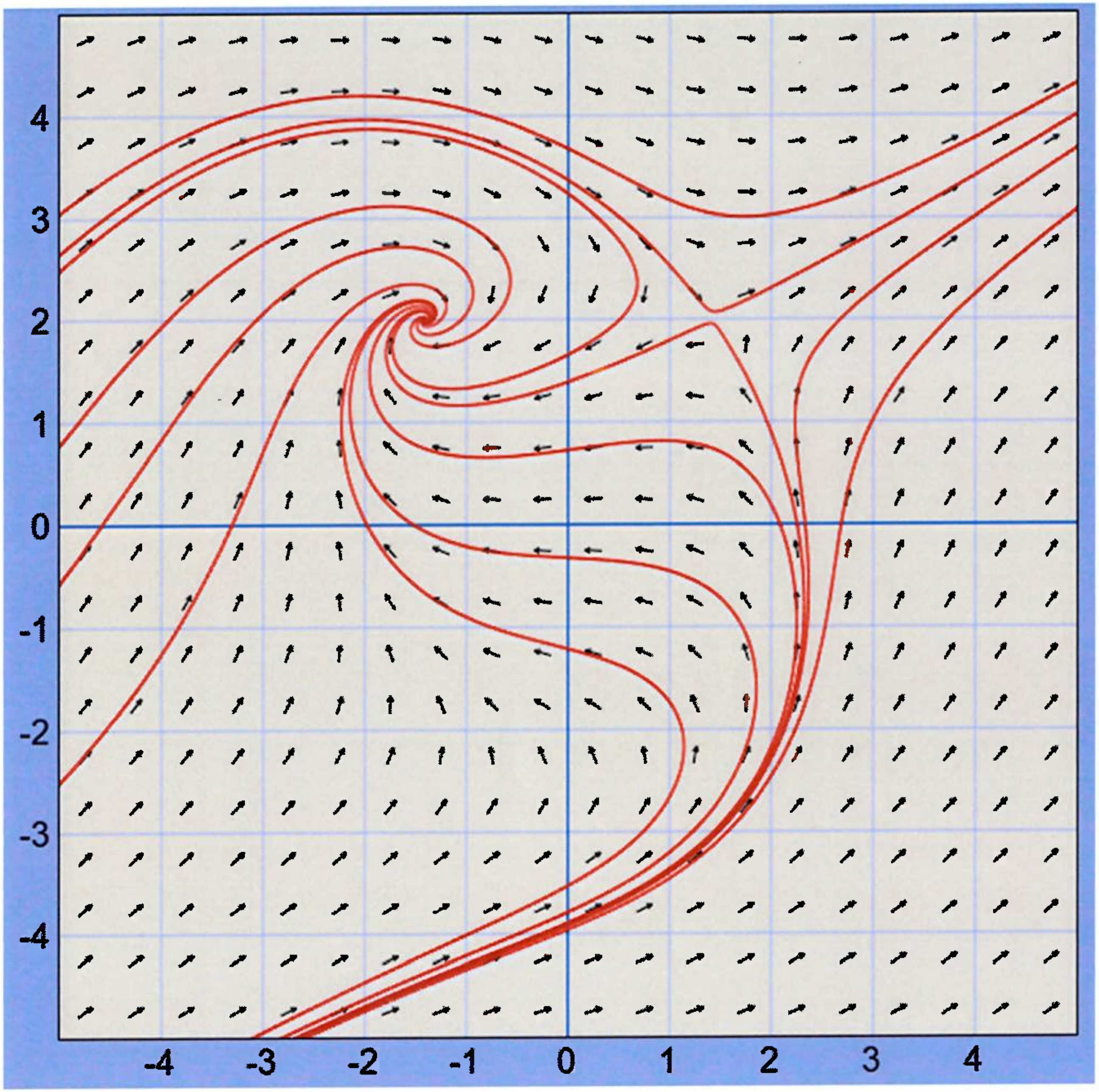
$J = \begin{bmatrix} 2\sqrt{2} & 4 \\ 2\sqrt{2} & -1 \end{bmatrix}$ system acts like $\vec{x}' = \begin{bmatrix} 2\sqrt{2} & 4 \\ 2\sqrt{2} & -1 \end{bmatrix} \vec{x}$




$\lambda \approx 4.8, -3$ Saddle point
unstable
(not sensitive to perturbation)

near $(-\sqrt{2}, 2)$

$J = \begin{bmatrix} -2\sqrt{2} & 4 \\ -2\sqrt{2} & -1 \end{bmatrix}$ $\lambda \approx -2 \pm 3.2i$ spiral sink
asymp. stable
(not sensitive to perturbation)

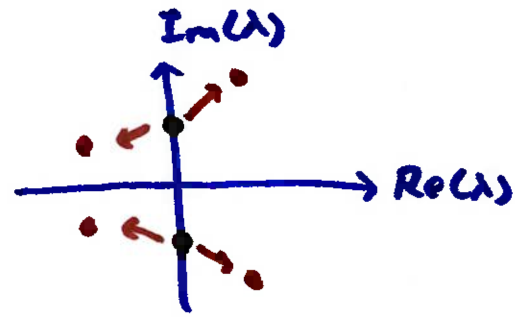

4
3
2
1
0
-1
-2
-3
-4
-4 -3 -2 -1 0 1 2 3 4

example

$x' = 2xy = F$

$y' = 1 - x^2 + y^2 = G$

cp: $(1,0)$, $(-1,0)$




$$J = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} 2y & 2x \\ -2x & 2y \end{bmatrix}$$

$$J(1,0) = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \qquad \lambda = \pm 2i$$

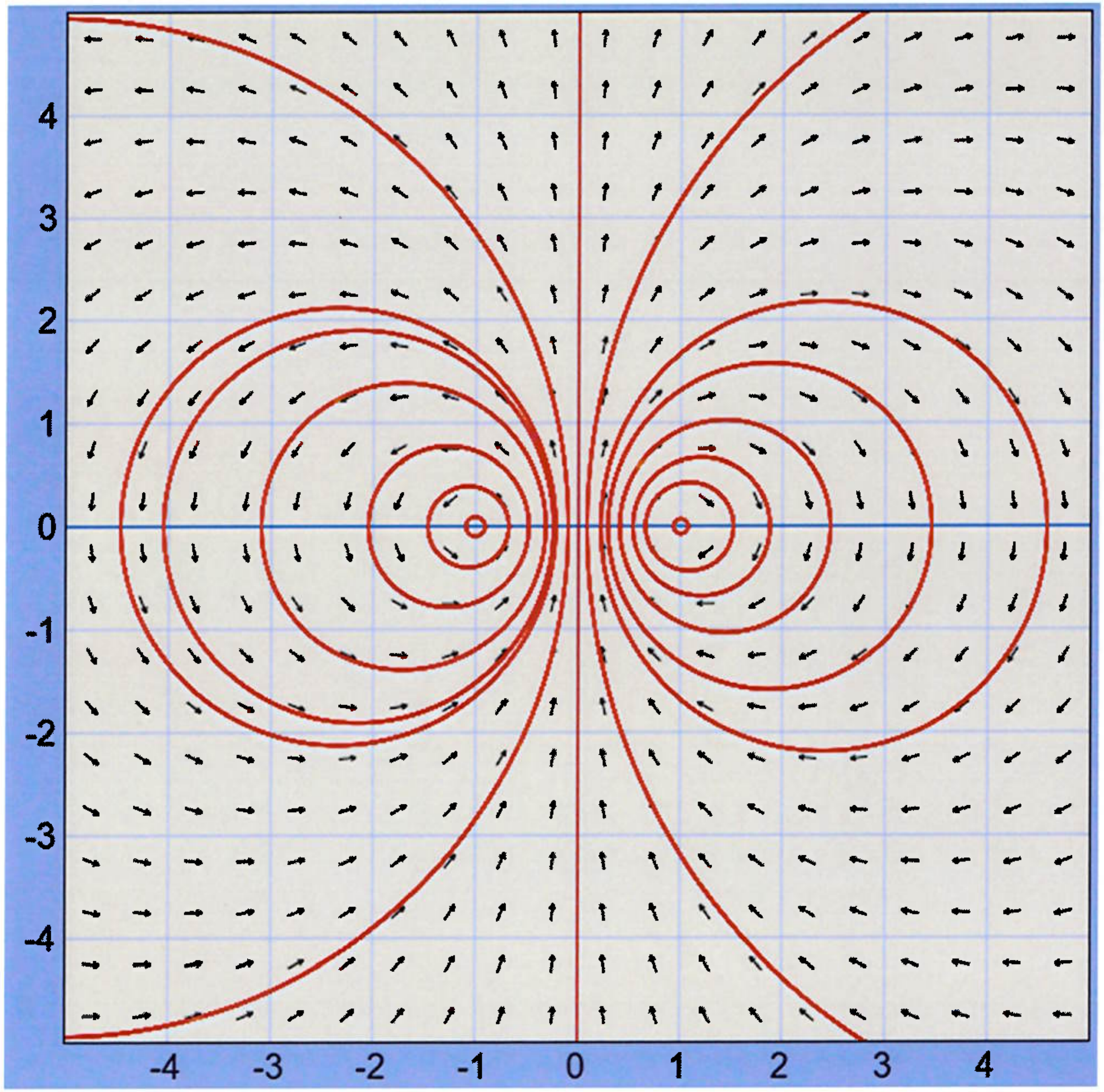
linearized sys. says center (stable) sensitive to perturbation true behavior could still be a center but we can't tell from linearization

$$J(-1,0) = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \qquad \lambda = \pm 2i$$

same story as the other cp.

we can't be sure what the true picture is like w/o either plotting the nonlinear phase diagram or solving the system (for this sys, the sub $u = y^2/x$ can solve the system)

in this case,
the linearized
sys tell us
the "truth"



<u>example</u>

$$x' = -3y + ay(x^2+y^2)$$

$$y' = 3x + ay(x^2+y^2)$$ a is some constant

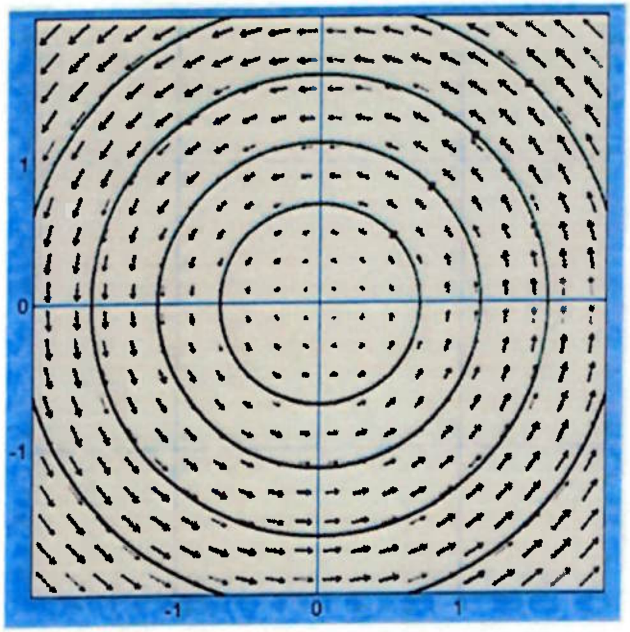
cp: $(0,0)$

$$J = \begin{bmatrix} 2xya & -3+ax^2+3y^2a \\ 3+2axy & 3ay^2+ax^2 \end{bmatrix}$$

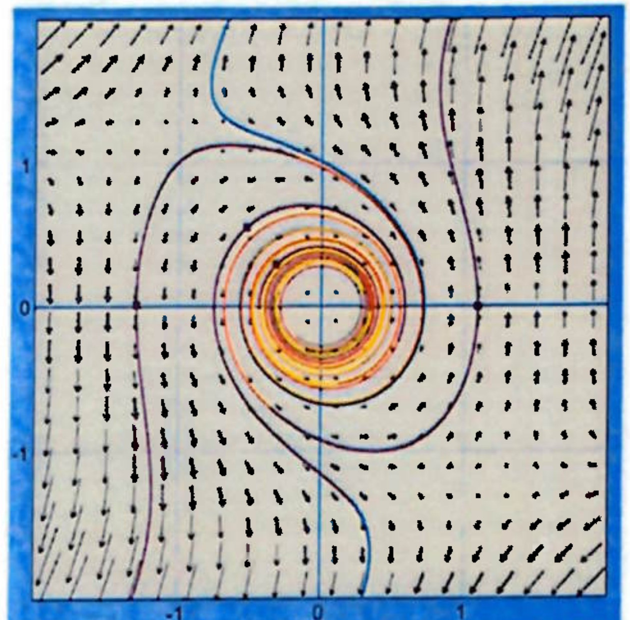
$$J(0,0) = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$$ no a here.

clearly "a" is doing something

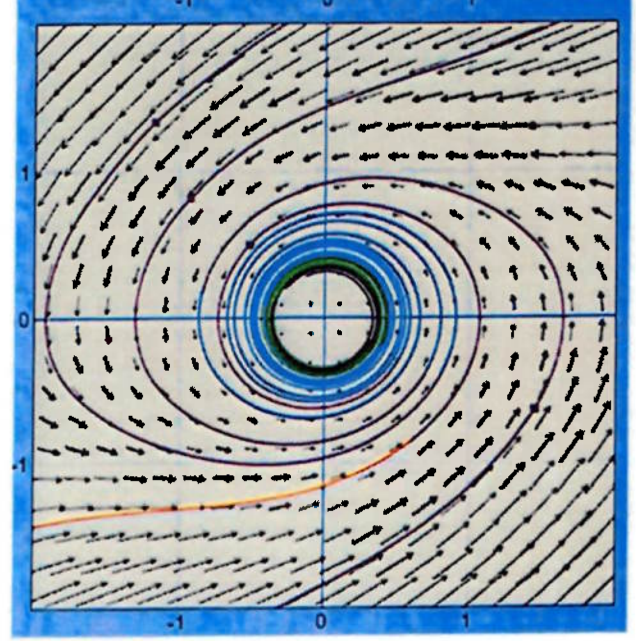
but linearization completely lost it



Linearized
(stable center)



Nonlinear a=1
(unstable
spiral point)



Nonlinear a= -1
(asymptotically
stable spiral
point)